SOLUTION BY J. A. CAPARO, University of Notre Dame.

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (I) and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ (II) be the equations of the hyperbola and its conjugate respectively. Let m be the slope of the set of parallel chords and $P_1(x_1y_1)$, $P_2(x_2y_2)$ be the points where the chord $y=mx+c$ intersects the primary and conjugate hyperbolas respectively; and $P(xy)$ the middle point between P_1 and P_2. Then

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \qquad \text{(I)}, \qquad \frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}=-1 \qquad \text{(II)},$$

$$y-y_1=m(x-x_1) \quad \text{(III)}, \qquad x=\frac{x_1+x_2}{2}, \qquad y=\frac{y_1+y_2}{2} \quad \text{(IV)}.$$

From II and IV, we have:

$$\frac{(2x-x_1)^2}{a^2}-\frac{(2y-y_1)^2}{b^2}=-1,$$

which becomes

$$\frac{x^2-xx_1}{a^2}-\frac{y^2-yy_1}{b^2}=-\frac{1}{2}.$$

Combining this equation with III we get,

$$x-x_1=\frac{a^2b^2}{2(a^2my-b^2x)};$$

from I and III we get,

$$\frac{x_1^2}{a^2}-\frac{[y-m(x-x_1)]^2}{b^2}=1.$$

Eliminating x_1 between these two equations we have

$$\frac{1}{a^2}\left[x-\frac{a^2b^2}{2(a^2my-b^2x)}\right]^2-\frac{1}{b^2}\left[y-\frac{a^2b^2m}{2(a^2my-b^2x)}\right]^2=1,$$

which, by I, reduces to

$$4\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(\frac{amy}{b}-\frac{bx}{a}\right)^2=(a^2m^2-b^2),$$

which is the general equation of the desired locus. As a particular case, if the chords are perpendicular to the x axis, then writing the equation in the form,

$$4\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(\frac{ay}{b}-\frac{bx}{am}\right)^2=a^2-\frac{b^2}{m^2}$$

and making $m=\infty$, we get

$$4b^2x^2y^2-4a^2y^4=a^2b^4.$$

Also solved by PAUL CAPRON.

CALCULUS.

393. Proposed by LAENAS G. WELD, Pullman, Illinois.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

SOLUTION BY FRANK R. MORRIS, Glendale, Calif.

Let the ellipse be represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

then each quadrant will contain one brick. Also let m be the length and n the thickness of a brick. The ellipse must pass through the point (m, n), i. e., the equation

$$\frac{m^2}{a^2}+\frac{n^2}{b^2}=1 \text{ must be true.}$$

From this we get

$$a = \frac{bm}{\sqrt{b^2 - n^2}}.$$

The area of the ellipse, which is easily found by integration, is πab. Substituting the above value of a in this expression we have

$$\frac{\pi b^2 m}{\sqrt{b^2 - n^2}},$$

which is a function, $f(b)$, of the independent variable b. To find the minimum value of $f(b)$ find the values of b which cause the first derivative to vanish.

$$f'(b) = \pi m \frac{2b(b^2 - n^2) - b^3}{(b^2 - n^2)^{3/2}} = 0.$$

Solving this equation we find b to be 0 or $n\sqrt{2}$. The latter value is obviously the desired one. The corresponding value of a is $m\sqrt{2}$.

Hence, the area is $\pi m\sqrt{2} \cdot n\sqrt{2}$ or $2\pi mn$.

From geometrical considerations it does not seem necessary to show that $f''(b)$ is positive and that, therefore, $f(b)$ is a true minimum.

Also solved similarly by H. S. Uhler, H. C. Feemster, George W. Hartwell, and H. L. Agard.

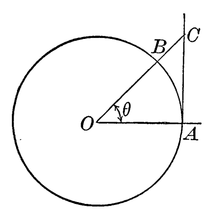
Note.—The above solution assumes that the bricks are laid side by side without mortar, whereas in a "brick wall" they are laid so as to *break joints*. Very possibly the question as answered is the one really intended by the proposer but the ellipse found certainly does not inclose four bricks as they are laid in a *brick wall*. Editors.

394. Proposed by W. W. BURTON, Macon, Ga.

A horse runs 10 miles per hour on a circular race-track in the center of which is an arc-light. How fast will his shadow move along a straight board fence (tangent to the track at the starting point) when he has completed one eighth of the circuit?

Solution by C. E. Horne, Westminster College, Colorado.

Let B and C be the position of the horse and shadow at any time after starting from the point of tangency, A. Let $AB = s$, $AC = y$, angle $AOC = \theta$, and $r =$ the radius of the ring. Then $s = r\theta$ (1) and $y = r \tan \theta$ (2). From (1), $ds/dt = r(d\theta/dt)$, the rate of the horse and from (2), $dy/dt = r \sec^2 \theta(d\theta/dt)$, the rate of the shadow, $= \sec^2 \theta(ds/dt) = 10 \sec^2 \theta = 20$ miles per hour when $\theta = \pi/4$.



Also solved by A. H. Holmes, H. L. Agard, J. A. Caparo, D. Rumble, W. C. Eells, Clifford N. Mills, Horace Olson, H. L. Agard, and George W. Hartwell.

395. Proposed by W. W. BURTON, Mercer University, Macon, Ga.

Into a full conical wine glass whose depth is a and whose angle at the base is 2α there is carefully dropped a spherical ball of such size as to cause the greatest overflow. Show that the radius of the ball is $a \sin \alpha/(\sin \alpha + \cos 2\alpha)$.

From Woods and Bailey's *A Course in Mathematics* (1907), Volume I, page 213.

Solution by H. S. Uhler, Yale University.

Let the sphere touch the inside of the cone near the rim with its center above the plane of the edge. Also, let h and r denote, respectively, the altitude of the submerged spherical segment and the radius of the sphere. Then $h = a + r(1 - \csc \alpha)$. The volume v of the liquid spilled is, of course, equal to the volume of the submerged spherical segment.

The volume of a spherical segment of altitude h, and radius of base r_1 is $\frac{1}{6}\pi h^3 + \frac{1}{2}\pi h r_1^2$. Here,

$$r_1 = \sqrt{2rh - h^2}, \qquad \text{hence} \qquad v = \pi h^2(r - \tfrac{1}{3}h).$$